# Announcements

1) Sections covered on the final:

1.1 - 1.4, 1.7, 1.10

2.1 - 2.3, 2.7, 3.1, 3.2,

4.1 - 4.8, 5.1 - 5.4,

6.1 - 6.3, 6.5, 6.6, 7.1

2) Applications guaranteed to be on the final:

PageRank, Circuits, Balancing chemical equations, Computer graphics, Signals, best fit lines/quadratics

3) Final replaces lowest test grade!

4) Math Colloquium

today, 3-4, CB 2046

"Intro to Multigrid Methods"

**Recall:** (least squares)

If $Ax=b$ has no solution, we can try to find "best-fit solutions" that minimize

$$\|Ax-b\|_2$$

## Equivalence ($A^tAx = A^tb$)

A least squares solution to $Ax = b$ satisfies

$$A^tAx = A^tb.$$

Moreover, if $A^tAx = A^tb$, then $x$ is a least-squares solution to $Ax = b$.

# Applications of Least-Squares

(Section 6.6)

Given two points $(x_0, y_0)$ and $(x_1, y_1)$ in $\mathbb{R}^2$, if the points are not identical, then there is a unique line passing through $(x_0, y_0)$ and $(x_1, y_1)$.

What if I have three points in $\mathbb{R}^2$?

**Example 1:** Is there a line passing through the points $(1,5)$ $(2,56)$, $(-1,11)$?

If the line passes through $(1,5)$ and $(2,56)$, then its slope is $51$.

The line through $(2,56)$ and $(-1,11)$ has slope $15 \neq 51$.

So there is no line that connects all three points!

The best we can do:

Find a line that minimizes the distances from all three points — this is a least-squares problem!

# Back to Example:

There is no line

$$y = mx + b$$

satisfying

$$5 = m + b$$

$$56 = 2m + b$$

and

$$11 = -m + b.$$

Rewrite as a matrix equation:

$$\begin{array}{c} \begin{array}{cc} \text{b coefficients} & \text{m coefficients} \end{array} \\ \begin{bmatrix} 1 & 1 \\ 1 & 2 \\ 1 & -1 \end{bmatrix} \end{array} \begin{bmatrix} b \\ m \end{bmatrix} = \begin{bmatrix} 5 \\ 56 \\ 11 \end{bmatrix}$$

With $A = \begin{bmatrix} 1 & 1 \\ 1 & 2 \\ 1 & -1 \end{bmatrix}$,

we are solving the

least-squares problem

$Ax = w$ where $w = \begin{bmatrix} 5 \\ 56 \\ 11 \end{bmatrix}$.

Same as actual solutions

to $$A^t A x = A^t w$$

$$A^t = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & -1 \end{bmatrix}$$

$$A^t w = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & -1 \end{bmatrix} \begin{bmatrix} 5 \\ 56 \\ 11 \end{bmatrix}$$

$$= \begin{bmatrix} 72 \\ 106 \end{bmatrix}$$

$$A^t A = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & -1 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 1 & 2 \\ 1 & -1 \end{bmatrix}$$

$$= \begin{bmatrix} 3 & 2 \\ 2 & 6 \end{bmatrix}$$

$$\det(A^t A) = 18 - 4 = 14 \neq 0$$

$$(A^tA)^{-1} = \frac{1}{14}\begin{bmatrix} 6 & -2 \\ -2 & 3 \end{bmatrix}$$

We then get

$$x = (A^tA)^{-1}A^t w$$

$$= \frac{1}{14}\begin{bmatrix} 6 & -2 \\ -2 & 3 \end{bmatrix}\begin{bmatrix} 72 \\ 106 \end{bmatrix}$$

$$= \frac{1}{14}\begin{bmatrix} 432-212 \\ -144+318 \end{bmatrix}$$

$$= \frac{1}{14}\begin{bmatrix} 220 \\ 174 \end{bmatrix} = \frac{1}{7}\begin{bmatrix} 110 \\ 87 \end{bmatrix}$$

This says

$$\boxed{\begin{aligned} m &= \frac{87}{7} \\ b &= \frac{110}{7} \end{aligned}}$$

gives the line that minimizes the distance from our three initial points.

# Best Fit Lines

(general)

Given points

$(x_1, y_1), (x_2, y_2), \ldots, (x_n, y_n)$

in $\mathbb{R}^2$. To find the

best fit line, find the

least-squares solution for

$m, b$ in

$$\text{n one's}\left\{\vphantom{\begin{bmatrix} 1 \\ 1 \\ \vdots \\ 1 \end{bmatrix}}\right.\underbrace{\begin{bmatrix} 1 & x_1 \\ 1 & x_2 \\ \vdots & \vdots \\ 1 & x_n \end{bmatrix}}_{A,\ \text{an } n\text{x}2 \text{ matrix}} \begin{bmatrix} b \\ m \end{bmatrix} = \begin{bmatrix} y_1 \\ y_2 \\ \vdots \\ y_n \end{bmatrix}$$

__Example 2:__ Set up a matrix equation for the best-fit line through $(2,-32)$, $(6,128)$, $(-\pi,11)$, $(15,56)$.

Equation:

$$\underbrace{\begin{bmatrix} 1 & 2 \\ 1 & 6 \\ 1 & -\pi \\ 1 & 15 \end{bmatrix}}_{A} \begin{bmatrix} b \\ m \end{bmatrix} = \begin{bmatrix} -32 \\ 128 \\ 11 \\ 56 \end{bmatrix}$$

The solution will be

$$(A^tA)^{-1}A^t\begin{bmatrix} -32 \\ 128 \\ 11 \\ 56 \end{bmatrix}$$

# Best-Fit Quadratics?!

Through any 3 points in $\mathbb{R}^3$, there is a unique quadratic equation that all three points satisfy.

If the points are

$(x_1, y_1)$, $(x_2, y_2)$, $(x_3, y_3)$

we want $a, b, c$ with

$$a x_1^2 + b x_1 + c = y_1$$

$$a x_2^2 + b x_2 + c = y_2$$

$$a x_3^2 + b x_3 + c = y_3$$

$(x_1 \neq x_2,\ x_2 \neq x_3,\ x_3 \neq x_1)$

Rewrite as a linear equation

$$\underbrace{\begin{bmatrix} 1 & x_1 & x_1^2 \\ 1 & x_2 & x_2^2 \\ 1 & x_3 & x_3^2 \end{bmatrix}}_{A} \begin{bmatrix} c \\ b \\ a \end{bmatrix} = \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix}$$

$\mathrm{rref}(A) = I_3$, so

there is always a solution for $a, b, c$!

If you have more than 3 points, you are again doing least squares, except now the equation is

$$\underbrace{\begin{bmatrix} 1 & x_1 & x_1^2 \\ 1 & x_2 & x_2^2 \\ \vdots & \vdots & \vdots \\ 1 & x_n & x_n^2 \end{bmatrix}}_{A} \begin{bmatrix} c \\ b \\ a \end{bmatrix} = \begin{bmatrix} y_1 \\ y_2 \\ \vdots \\ y_n \end{bmatrix}$$

solve the same way as for lines!